

LEFT SYMMETRIC ALGEBRAS FROM DNA INSERTION

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ABSTRACT. In this paper, we construct some left-symmetric algebras arising from the operation of DNA insertion. We define a new operation of insertion by modifying the simplified insertion as follows: $x \Rightarrow y := f(|x|, |y|) \sum_{i=0}^q y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q$ where $x = x_1 x_2 \cdots x_p$ and $y = y_1 y_2 \cdots y_q$. Then it is proved that $(x, y, z)_2 = (y, x, z)_2$ if and only if f satisfies the following conditions:

$$f(m, n)f(m + n, p) = f(n, p)f(m, n + p) = f(m, p)f(n, m + p)$$

where (x, y, z) is defined as (4.1) and $m, n, p \in \mathbb{N}$. The paper also give an important example: $f(m, n) = \exp\{g(m, n)\}$, where $g(m, n) = k \cdot mn$, k is a fixed positive number. We also give simple case to make clear that the intermediate version of the insertion operation which is the open problem numbered [5] and the operator of synchronized insertion given in the [2] by Bremner do not satisfy the left-symmetric identities.

1. INTRODUCTION

The genetic information is realized by the DNA recombination. The recombination provides the genetic program of development and functioning of all living organisms. The algebraic formalization of DNA recombination is represented in the form of linear space $F(R)$ over a field F of characteristic 0, where R is an infinite free semigroup generated by the set of DNA nucleotides $\{A, G, C, T\}$. The schematic model of non-homologous DNA recombination can be represented in the form: $(ab) \cdot c \rightarrow acb$, where two chromosomes (ab) and c are participating in non-homologous recombination. The algebraic formalization of the above recombination, with respect of all possible insertions DNA c in DNA (ab) , defines the algebra of simplified insertions. Let $X \in R$. Then X can be presented as $X = x_1 x_2 \cdots x_n$, where $x_i \in \{A, G, C, T\}$. The operation, defined in [10], is given by

$$(1.1) \quad X \cdot a = \sum_{i=0}^n x_1 \cdots x_i a x_{i+1} \cdots x_n,$$

where a is an arbitrary element in $F(R)$; and $b.a = \sum \alpha_s (u_s \cdot a)$, where $b = \sum_s \alpha_s u_s$, $\alpha_s \in F$, and u_s are monomials in R . This operation \cdot is called the operation of right simplified insertion. The operation of simplified insertion was firstly introduced by M. Bremner [2], and it is an algebraic formalization of the operation of normal insertion in the theory of DNA computing (see [6]). This algebra defined on $F(R)$ are usual non-associative algebras. Nonassociative algebras have previously been applied to population genetics in the well-developed theory of genetic algebras. For surveys of this area see Reed [9].

The operation of the algebra $F(R)$ defined by (1.1) is called right simple insertion. It satisfies the left-symmetric identity $(x, y, z) = (y, x, z)$. Left-symmetric algebras (also called pre-Lie algebras, or Koszul-Vinberg algebra) were originally introduced by A. Cayley in 1896, in the context of rooted tree algebras, see the A. Cayley [4]. Vinberg [11] and

Koszul [8] introduced them in the early 1960s in connection with convex homogeneous cones and affinely flat manifolds. A few years later, they also appeared naturally in the theory of cohomology of associative algebras (Gerstenhaber [6]). They have also been studied from a purely algebraic point of view (Kleinfeld [7]).

The right simple insertion left symmetric algebra is infinite-dimensional. However, one find there are only finite types DNA in nature. So one must change the operation of the simple right insertion. We change the definition of right simple insertion by some coefficient as following:

$$(1.2) \quad (x_1 \cdots x_n) \cdot (y_1 \cdots y_m) = e^{kmn} \sum_{i=0}^n x_1 \cdots x_k y_1 \cdots y_m x_{k+1} \cdots x_n.$$

When the length of genetics becomes larger and $k < 0$, the function becomes smaller. So, if the length of genetics is very large, the results of new genetics, generated by the simple right insertion, is killed by this coefficient. Fortunately, $F(R)$ with the operation given by (1.2) is still a left symmetric algebra. Moreover, we define a new operation of insertion as follows

$$x \Rightarrow y := f(|x|, |y|) \sum_{i=0}^q y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

where $x = x_1 x_2 \cdots x_p$ and $y = y_1 y_2 \cdots y_q$. Then $F(R)$ becomes a left symmetric algebra if and only if $f(m, n)$ satisfies the following conditions:

$$f(m, n)f(m+n, p) = f(n, p)f(m, n+p) = f(m, p)f(n, m+p),$$

where $m, n, p \in \mathbb{N}$.

The paper is organized as follows. In section (II), we recall the constructed left-symmetric algebras from Simplified insertion. In section (III), we recall the operations of synchronized insertion and simplified insertion on words, and give simple that Synchronized insertion does not satisfy left symmetric identity, but simplified insertion satisfies the left-symmetric identity; In section (IV), we give another definition of simplified insertion and construct new left symmetric structure from this definition of simplified insertion and discuss some properties of these left-symmetric algebras and certain application in genetics and molecular genetics.

2. DEFINITIONS AND NOTIONS

2.1. The left symmetric algebra. Let F be a field of characteristic zero. Let A be an algebra over F . For elements x and y in A the bilinear product is denoted by $(x, y) \rightarrow x \circ y$. Given three elements x, y and z , we will denote by (x, y, z) the associator of these elements by

$$(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z).$$

Definition 2.1. A left-symmetric algebra is an algebra whose associator is left-symmetric, that is

$$\forall x, y, z \in L, \text{ or equivalently } (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

Example 2.1.

- (a) Every associative algebra is a left symmetric algebra with the left symmetric structure given by

$$x \circ y := xy,$$

Obviously, $(x \circ y) \circ z - x \circ (y \circ z) = 0$.

- (b) Let A be the vector space $C^\infty(R, R)$ of smooth functions. For f and g in A we set $fg = f \frac{dg}{dx}$.

2.2. Languages and Simplified insertion. We denote S a finite non-empty set. A word over S is a finite string $w = a_1 a_2 \cdots a_p$ where $p \geq 0$ and $a_i \in S$ ($1 \leq i \leq p$). For $p = 0$ we have the empty word denoted 1. We denote the length of w by $|w| = p$. We write $M(S)$ for the set of all words; any subset of $M(S)$ is called a language.

We now consider the operation of DNA insertion of one word into another. This gives a method of combination of DNA molecules which are well-studied in molecular genetics. Given $x, y \in M(S)$ we consider all insertions of x into y :

$$x \rightarrow y = \{y^1 x y^2 \mid y = y^1 y^2, y^1, y^2 \in M(S)\}.$$

Note that we are allowing both y^1 and y^2 to be empty.

Firstly, we recall the simplified insertion.

Definition 2.2. The simplified insertion of x into y is the linearized form of this operation, that is the sum of all insertions:

$$x \rightarrow y = \sum_{i=0}^q y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

where $y = y_1 y_2 \cdots y_q$, $y_j \in S$ and $x = x_1 x_2 \cdots x_p$, $x_j \in S$.

Given a free associative algebra $A[X]$ on a set of generators $S = \{a_1, a_2, \dots, a_s\}$. we can define a new operation of multiplication \circ by the following rule:

$$(2.1) \quad x \circ y = x \rightarrow y = \sum_{i=0}^q y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

where $y = y_1 y_2 \cdots y_q$, $y_j \in S$. The associator for this new operation, that is simplified insertion operation, is defined as usual by

$$(2.2) \quad (x, y, z)_1 = (x \circ y) \circ z - x \circ (y \circ z) = (x \rightarrow y) \rightarrow z - x \rightarrow (y \rightarrow z).$$

Example 2.2.

- (a) Let $x = a_1 a_2 a_3$, $y = a_4 a_5$, then $x \rightarrow y = a_1 a_2 a_3 a_4 a_5 + a_4 a_1 a_2 a_3 a_5 + a_4 a_5 a_1 a_2 a_3$

The following Theorem 2.1 was first proved by Gerstenhaber [6].

Theorem 2.1. The algebra $A[X]$ is a free left symmetric algebra satisfies the left symmetric identities $(x, y, z) = (y, x, z)$, where (x, y, z) is defined by (2.2). It is a left symmetric algebra.

The following is just like the proof of Bremner [2].

Proof. we just prove that

$$(2.3) \quad (x \rightarrow y) \rightarrow z - x \rightarrow (y \rightarrow z) = (y \rightarrow x) \rightarrow z - y \rightarrow (x \rightarrow z).$$

Let

$$x = x_1 x_2 \cdots x_p, \quad y = y_1 y_2 \cdots y_q, \quad z = z_1 z_2 \cdots z_r.$$

Using (2.1), we obtain

$$x \rightarrow y = \sum_{j=0}^{j=q} y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q,$$

$$y \rightarrow z = \sum_{k=0}^{k=r} z_1 z_2 \cdots z_k y z_{k+1} \cdots z_r$$

Then

$$\begin{aligned} & x \rightarrow (y \rightarrow z) \\ = & \sum_{k=0}^r \left\{ \sum_{j=0}^{k-1} z_1 z_2 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r + z_1 z_2 \cdots z_k (x \rightarrow y) z_{k+1} \cdots z_r \right. \\ & \left. + \sum_{j=k+1}^r z_1 z_2 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r \right\}. \end{aligned}$$

$$(2.4) \quad (x \rightarrow y) \rightarrow z = \sum_{j=0}^q \sum_{k=0}^r z_1 z_2 \cdots z_k y_1 y_2 \cdots y_i x y_{j+1} \cdots y_q z_{k+1} \cdots z_r$$

Using 4.4 and 2.4, we have

$$\begin{aligned} & (x \rightarrow y) \rightarrow z - x \rightarrow (y \rightarrow z) \\ = & - \sum_{k=0}^r \left\{ \sum_{j=0}^{k-1} z_1 z_2 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r - \sum_{j=k+1}^r z_1 z_2 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r \right\} \\ (2.5) \quad & LHS \end{aligned}$$

$$\begin{aligned} & (y \rightarrow x) \rightarrow z - y \rightarrow (x \rightarrow z) \\ = & - \sum_{k=0}^r \left\{ \sum_{j=0}^{k-1} z_1 z_2 \cdots z_j y z_{j+1} \cdots z_k x z_{k+1} \cdots z_r - \sum_{j=k+1}^r z_1 z_2 \cdots z_k x z_{k+1} \cdots z_j y z_{j+1} \cdots z_r \right\} \\ (2.6) \quad & RHS \end{aligned}$$

The only terms of *LHS* which remain are those in which x and y are both inserted into z and are separated by at least one letter of z from (2.5). We immediately obtain (2.3). \square

Remark 2.1. We consider the following open problem numbered [5] given in the [2] by Bremner.

Recall the open problem [5]: An intermediate version of the insertion operation is obtained when we regard the set-theoretic definition as producing a set rather than a multiset. That is, on words $x = x_1 \cdots x_p$ and $y = y_1 \cdots y_q$ with $x_i, y_j \in S$ we define

$$\delta(x, y) = \begin{cases} 1, & x_1 = y_1 \\ 0, & \text{otherwise} \end{cases}$$

We then consider the operation

$$x \circ y = \sum_{j=0}^q \delta(x, y_j \cdots y_q) y_1 y_2 \cdots y_{j-1} x y_j \cdots y_q$$

What are the polynomial identities satisfied by this operation? We only check the left-symmetry identities of degree 3.

Choose $S = \{a, b, c\}$, $x = ab$, $y = abc$, $z = ac$. Then $x \circ y = ababc$, $(x \circ y) \circ z = ababcac$, $x \circ (y \circ z) = ababcac + abcabac$ and hence $(x \circ y) \circ z - x \circ (y \circ z) = -abcabac$. Similarly, $(y \circ x) \circ z - y \circ (x \circ z) = -ababcac$. It follows that it does not satisfy the left-symmetric identity $(x, y, z) = (y, x, z)$.

Remark 2.2. Recall another DNA insertion operation the linearized version of synchronized insertion introduced by Bremner in the [2]. We show that it does not satisfy the left-symmetric identity $(x, y, z) = (y, x, z)$, either.

$$x \rightrightarrows y = \sum_{j=1}^q s(x, y_j \cdots y_q) y_1 y_2 \cdots y_{j-1} x y_j \cdots y_q$$

where $s(x, y)$ is given by:

$$s(x, y) = k \text{ when } x_i = y_i \text{ for } 1 \leq i \leq k \text{ but } x_{k+1} = y_{k+1} \text{ (or } k+1 \geq \min(p, q))$$

Without loss of generality, we choose the simplest case of the linearized version of synchronized insertion in which S contains one letter: $S = \{a\}$. Then the synchronized insertion becomes

$$a^p \rightrightarrows a^q = c(p, q) a^{p+q}$$

where

$$c(p, q) = \begin{cases} \frac{1}{2}p(2q - p + 1) & p < q \\ \frac{1}{2}q(q + 1) & p \geq q \end{cases}$$

Choose $p = 2, q = 3, r = 6$, then $(x \circ y) \circ z - x \circ (y \circ z) - (y \circ x) \circ z - y \circ (x \circ z) = 16 \neq 0$. Hence we show that it does not satisfy the left-symmetric identity.

3. ONE OF THE SUBSPACES AND LEFT SYMMETRIC ALGEBRA

Firstly, we describe how signs may be introduced in the notions of words and DNA insertions. We will use the notation $\delta(x, y)$ for the sign introduced when x and y are both in a word, i.e., $\delta(x, y) = 1$ if there is some words like $\cdots xy \cdots$ and 0 otherwise. Suppose that $S = a_1, a_2, \cdots, a_p$ and $\delta(a_i, a_j) = 1$, for $\forall a_i, a_j \in S$.

$$\delta(a_i, a_j) = \begin{cases} 1, & a_i, a_j \text{ can be connected} \\ 0, & \text{otherwise} \end{cases}$$

Consider a subspace of the $A[X]$, $\exists a_i, a_j$ s.t. $\delta(a_i, a_j) = 0$. That is no any word containing $a_i a_j$.

We give an example:

Example 3.1.

(a) $S = \{a, b, c\}, \delta(a, b) = \delta(b, c) = \delta(a, c) = 1$. We can use the diagram $\begin{smallmatrix} \circ & \circ & \circ \\ a & b & c \end{smallmatrix}$ and $\begin{smallmatrix} \circ & \circ \\ a & c \end{smallmatrix}$. Set $x = abc, y = bc$, then $x \circ y = abcbc + babcc + bcabc$.

(b) $S = \{a, b, c\}, \delta(a, b) = \delta(b, c) = 1, \delta(a, c) = 0$ We can use the diagram $\underset{a}{\circ} - \underset{b}{\circ} - \underset{c}{\circ}$ and there is no $-$ between a and c . Set $x = abc, y = bc$, then $x \circ y = abcbc + babcc$.

We consider $\delta(a_i, a_i) = 1, \forall a_i \in S$.

Theorem 3.1. $S = \{a_1, \dots, a_p\}, \|S\| = p$

- (a) $\|S\| = 1$, the operation \circ satisfies the left symmetric identities .
- (b) $\|S\| = 2$ and $\delta(a_1, a_2) = 1$, the operation \circ satisfies the left symmetric identities.
 $\|S\| = 2$ and $\delta(a_1, a_2) = 0$, the operation \circ is associative. It is left symmetric algebra obviously.
- (c) $\|S\| \geq 3$ and $\exists a_i, a_j \in S, s.t. \delta(a_i, a_j) = 0$, The operation \circ is non-associative and non-commutative. $A[X]$ is not left symmetric algebra. Otherwise, the operation \circ is non-associative and non-commutative. It satisfies however the left identities.

Proof. The first two items (a) and (b) follow directly from the theorem 2.1. Obviously $\|S\| = 2$ and $\delta(a_1, a_2) = 0$, the operation \circ satisfies $(a_i \circ a_j) \circ a_k = a_i \circ (a_j \circ a_k) = 0$ for $a_i, a_j, a_k \in \{a_1, a_2\}$. Hence we have (a) and (b).

It is not difficult to prove that for $n = 3$ the claims are true particularly.

(c) follows since the space of $n = 3$ is the subspace of $n \geq 3$. □

Example 3.2. $S = \{a, b, c\}, n = 3, \delta(a, b) = \delta(b, c) = 1, \delta(a, c) = 0$. Set $x = a, y = b, z = c$, then $x \circ y = ab + ba$ and $y \circ z = bc + cb$.
then

$$\begin{aligned}
 (x \circ y) \circ z &= abc + cba, (y \circ x) \circ z = abc + cba \\
 x \circ (y \circ z) &= abc + cba, y \circ (x \circ z) = 0. \\
 LHS &:= (x \circ y) \circ z - x \circ (y \circ z) = 0. \\
 (3.1) \quad RHS &:= (y \circ x) \circ z - y \circ (x \circ z) = abc + cba
 \end{aligned}$$

4. THE MODIFIED SIMPLIFIED INSERTION

4.1. The definition of modified simplified insertion. We can modify the simplified insertion as follows.

$$x \Rightarrow y := f(|x|, |y|) \sum_{i=0}^q y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q$$

where the function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ is given by $(m, n) \mapsto f(m, n)$

Then we have

$$(4.1) \quad (x, y, z)_2 = (x \circ y) \circ z - x \circ (y \circ z) = (x \Rightarrow y) \Rightarrow z - x \Rightarrow (y \Rightarrow z).$$

Example 4.1. Let $x = a_1 a_2 a_3, y = a_4 a_5, x \Rightarrow y = f(3, 2)(a_1 a_2 a_3 a_4 a_5 + a_4 a_1 a_2 a_3 a_5 + a_4 a_5 a_1 a_2 a_3)$.

4.2. The left symmetric algebra for modified simplified insertion.

Theorem 4.1. $A[X]$ is the left symmetric algebra where (x, y, z) is defined as (4.1) if and only if f satisfies the following conditions:

$$f(m, n)f(m + n, p) = f(n, p)f(m, n + p) = f(m, p)f(n, m + p)$$

where $m, n, p \in \mathbb{N}$.

Proof. As the same as Theorem 2.1, we are just to prove

$$(4.2) \quad (x \Rightarrow y) \Rightarrow z - x \Rightarrow (y \Rightarrow z) = (y \Rightarrow x) \Rightarrow z - y \Rightarrow (x \Rightarrow z).$$

Let

$$x = x_1 x_2 \cdots x_p, \quad y = y_1 y_2 \cdots y_q, \quad z = z_1 z_2 \cdots z_r.$$

We obtain

$$x \Rightarrow y = f(|x|, |y|) \sum_{j=0}^q y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q,$$

$$y \Rightarrow z = f(|y|, |z|) \sum_{k=0}^r z_1 z_2 \cdots z_k y z_{k+1} \cdots z_r$$

Then

$$(4.3) \quad \begin{aligned} & x \Rightarrow (y \Rightarrow z) \\ &= f(|y|, |z|) f(|x|, |y| + |z|) \sum_{k=0}^r \left\{ \sum_{j=0}^{k-1} z_1 z_2 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r \right. \\ & \quad \left. + z_1 z_2 \cdots z_k (x \rightarrow y) z_{k+1} \cdots z_r + \sum_{j=k+1}^r z_1 z_2 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r \right\}. \end{aligned}$$

$$(4.4) \quad \begin{aligned} & (x \Rightarrow y) \Rightarrow z \\ &= f(|x|, |y|) f(|x| + |y|, |z|) \sum_{j=0}^q \sum_{k=0}^r z_1 \cdots z_k y_1 \cdots y_j x y_{j+1} \cdots y_q z_{k+1} \cdots z_r \end{aligned}$$

Denote $H(|x|, |y|, |z|)$ and $H_2(|x|, |y|, |z|)$ as

$$f(|x|, |y|) f(|x| + |y|, |z|) - f(|y|, |z|) f(|x|, |y| + |z|)$$

and

$$f(|y|, |z|) f(|x|, |y| + |z|),$$

respectively. Using 4.3 and 4.4, we obtain

$$(4.5) \quad \begin{aligned} & (x \Rightarrow y) \Rightarrow z - x \Rightarrow (y \Rightarrow z) \\ &= H(|x|, |y|, |z|) \sum_{k=0}^r \sum_{j=0}^q z_1 \cdots z_k y_1 \cdots y_j x y_{j+1} \cdots y_q z_{k+1} \cdots z_r \\ & \quad - H_2(|x|, |y|, |z|) \sum_{k=0}^r \sum_{j=0}^{k-1} z_1 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r \\ & \quad - H_2(|x|, |y|, |z|) \sum_{k=0}^r \sum_{j=k+1}^r z_1 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r \\ &:= LHS \end{aligned}$$

$$(y \Rightarrow x) \Rightarrow z - y \Rightarrow (x \Rightarrow z)$$

$$\begin{aligned}
&= H(|y|, |x|, |z|) \sum_{k=0}^r \sum_{j=0}^p z_1 \cdots z_k x_1 \cdots x_j y x_{j+1} \cdots x_q z_{k+1} \cdots z_r \\
&\quad - H_2(|y|, |x|, |z|) \sum_{k=0}^r \sum_{j=0}^{k-1} z_1 \cdots z_j y z_{j+1} \cdots z_k x z_{k+1} \cdots z_r \\
&\quad - H_2(|y|, |x|, |z|) \sum_{k=0}^r \sum_{j=k+1}^r z_1 \cdots z_k x z_{k+1} \cdots z_j y z_{j+1} \cdots z_r \\
(4.6) \quad &:= RHS
\end{aligned}$$

Since the beginning term of LHS is yxz with the coefficient $H(|x|, |y|, |z|)$ and there is no other terms containing yxz . The remaining terms are those in which x and y are both inserted into z and are separated by at least one letter of z from (4.5).

Therefore, for every x, y, z , $LHS = RHS$ if and only if $H(|x|, |y|, |z|) = H(|y|, |x|, |z|) = 0$ and $H_2(|x|, |y|, |z|) = H_2(|y|, |x|, |z|)$.

We immediately obtain (4.2). \square

Remark 4.1. If we denote $H_1(|x|, |y|, |z|)$ as $f(|x|, |y|)f(|x| + |y|, |z|)$, then we have

$$H_1(|x|, |y|, |z|) = H_2(|x|, |y|, |z|), \quad H_2(|x|, |y|, |z|) = H_2(|y|, |x|, |z|).$$

That is

$$f(|x|, |y|)f(|x| + |y|, |z|) = f(|y|, |z|)f(|x|, |y| + |z|)$$

and

$$f(|y|, |z|)f(|x|, |y| + |z|) = f(|x|, |z|)f(|y|, |x| + |z|).$$

4.3. The existence of f . Recall that f satisfies the following equations

$$(4.7) \quad f(m, n)f(m + n, p) = f(n, p)f(m, n + p)$$

$$(4.8) \quad f(n, p)f(m, n + p) = f(m, p)f(n, m + p)$$

where $\forall m, n, p \in \mathbb{N}$.

Example 4.2. We choose a symmetric bilinear function $g: \mathbb{N} \times \mathbb{N} \rightarrow R$ given by $(m, n) \mapsto g(m, n)$. Hence $g(m, n) = k \cdot mn$, k is a fixed positive number.

Let $f(m, n) = \exp\{g(m, n)\}$, then

$$\begin{aligned}
&f(m, n)f(m + n, p) \\
&= \exp\{g(m, n)\} \exp\{g(m + n, p)\} \\
&= \exp\{g(m, n) + g(m + n, p)\} \\
&= \exp\{g(m, n) + g(m, p) + g(n, p)\} \\
&= \exp\{g(m, n + p) + g(n, p)\} \\
&= \exp\{g(m, n + p)\} \exp\{g(n, p)\} \\
(4.9) \quad &= f(m, n + p)f(n, p) \\
&f(n, p)f(m, n + p) \\
&= \exp\{g(n, p)\} \exp\{g(m, n + p)\}
\end{aligned}$$

$$\begin{aligned}
&= \exp\{g(n, p) + g(m, p) + g(n, p)\} \\
&= \exp\{g(m, n) + g(m, p) + g(n, p)\} \\
&= \exp\{g(n, m + p) + g(m, p)\} \\
&= \exp\{g(n, m + p)\} \exp\{g(m, p)\} \\
(4.10) \quad &= f(n, m + p)f(m, p)
\end{aligned}$$

then f satisfies (4.7) and (4.8).

Example 4.3.

- (a) $f(m, n) \equiv C_0$, C_0 is a constant. It is the same as the simplified insertion.
(b) We can easily check that f satisfies the (4.7) and (4.8) where f is defined as follows:

$$(4.11) \quad f(m, n) = \begin{cases} 1, & m, n \text{ are both odds} \\ 0, & \text{otherwise} \end{cases}$$

4.4. The symmetric of f . Let $n = p$ in the (4.7) and (4.8), then we get $f(m + p, p) = f(p, m + p)$, for $m \in \mathbb{N}$, then

$$f(m, n) = f(n, m) \text{ for } m, n \in \mathbb{N},$$

REFERENCES

- [1] M. Aguiar, Infinitesimal bialgebras, pre-Lie and dendriform algebras, Hopf algebras, pages 1C33, Lecture Notes in Pure and Applied Mathematics 237, Marcel Dekker, 2004. MR2051728 (2005c:16053)
- [2] M. R. Bremner, DNA computing, insertion of words and left-symmetric algebras, Proc. of the Maple Conf. 2005. Maple Inc. Waterloo. 2005.
- [3] M. R. Bremner, Additive structure of free left-symmetric and assosymmetric rings, International Journal of Mathematics, Game Theory and Algebra 12, 1 (2002) 23C37. MR1904877 (2003b:17004)
- [4] A. Cayley, On the Theory of Analytic Forms Called Trees. Collected Mathematical Papers of Arthur Cayley, Cambridge Univ. Press, Cambridge, 1890, Vol. 3 (1890), 242-246.
- [5] M. Daley, L. Kari, I. McQuillan: Families of languages defined by ciliate bio-operations. Theoret. Comput. Sci. 2004. V. 320. N. 1. P. 51-69.
- [6] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78(1963), 267-288.
- [7] E. Kleinfeld, On rings satisfying $(x, y, z) = (x, z, y)$, Algebras Groups and Geometries 4, 2 (1987) 129C138. MR914169 (89a:17001).
- [8] J-L. Koszul Domaines bornes homogenes et orbites de groupes de transformations affines, Bulletin de la Societe Mathematique de France 89 (1961), p. 515-533.
- [9] Mary Lynn Reed, Algebraic structure of genetic inheritance, Bulletin of the American Mathematical Society 34, 2 (1997) 107-130. MR1414973 (98e:17043)
- [10] SERGEI R. SVERCHKOV, ALGEBRAIC THEORY OF DNA RECOMBINATION.
- [11] E.B. Vinberg, Convex homogeneous cones, Transl. Moscow Math. Soc. 12 (1963), 340-403.

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